very bad. The non-defective cases, with several vectors nearly parallel, were sometimes worse than the defective cases.

$N = 6 \quad s = -4$

actual roots	0,	0	2,	2	−1	−4	−10	−18
approximations (7070)	.0228	−.0225	1.995,	2.004	−1	−4	−10	−18

$N = 6 \quad s = -6$

actual roots	−6	0, 0	4, 4	6, 6
approximations (7070)	−6	$-.0056 \pm .111i$	$3.969 \pm .363i$	$6.036 \pm .2215i$

$N = 6 \quad s = -6.5$

actual roots	−3	0	2.5	4.5	6	7, 7.75
approximations (7070)	−3	.0066	2.436	5.103	5.151	$7.401 \pm .405i$

The effect of higher precision is seen in the following example which was run on both the 7070 (8 digits) and the 1604 (10 digits)

$N = 10 \quad s = -14$

actual roots	approximations (CDC 1604)	approximations (IBM 7070)
0	-2×10^{-7}	.0026
12	12.00004	11.898
22	21.995	20.195
30, 30	$29.638 \pm 1.134i$	$23.843 \pm 7.16i$
36, 36	$34.993 \pm 2.94i$	$33.926 \pm 12.92i$
40, 40	$40.024 \pm 3.39i$	36.977
42, 42	$43.347 \pm 1.49i$	$46.387 \pm 10.99i$
		52.594

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Multivariate Polynomial Approximation for Equidistant Data

By B. Mond

Abstract. The theory of polynomial approximation for evenly spaced points is extended to multivariate polynomial approximation. It is also shown how available tables prepared for univariate approximation can be used in the multivariate case.

1. Introduction. Assume $f(x)$ is given for $x = x_1, x_2, \cdots, x_n$ and it is desired to approximate $f(x)$ by a polynomial of degree p, $1 \leqq p < n$, i.e.

$$f(x) \approx \sum_{i=0}^{p} a_i x^i.$$

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In order to determine the coefficients a_i so as to minimize

$$\sum_{j=1}^{n}\left[f(x_j) - \sum_{i=0}^{p} a_i x_j^{i}\right]^2$$

one must solve a set of $p + 1$ normal equations. If the x_i are evenly spaced, the problem can be greatly simplified by a change of variable and the use of orthogonal polynomials [4]. This simplification will now be extended to approximation by multivariate polynomials.

2. Bivariate Polynomial Approximation. Assume that $f(x, y)$ is defined on a finite planar set of points $\{(x_i, y_j)\}$ $(i = 1, 2, \cdots, n; j = 1, 2, \cdots, m)$ and that it is desired to approximate $f(x, y)$ by a bivariate polynomial of the form

$$\sum_{h=0}^{s}\sum_{k=0}^{r} a_{kh} x^k y^h$$

$1 \leqq r < n$ and $1 \leqq s < m$. In order to determine the coefficients a_{kh} $(k = 0, 1, \cdots, r; h = 0, 1, \cdots, s)$ so as to minimize

$$\sum_{j=1}^{m}\sum_{i=1}^{n}\left[f(x_i, y_j) - \sum_{h=0}^{s}\sum_{k=0}^{r} a_{kh} x_i^k y_j^h\right]^2$$

one must solve a system of $(r + 1)(s + 1)$ normal equations.

Assume, now, that the x_i and y_j are evenly spaced, i.e.

$$x_{k+1} - x_k = h_1 \qquad (k = 1, 2, \cdots, n - 1)$$

$$y_{k+1} - y_k = h_2 \qquad (k = 1, 2, \cdots, m - 1).$$

Thus, if we write

$$\begin{aligned} x &= a + ih_1 \\ y &= b + jh_2, \end{aligned} \tag{1}$$

where $a = x_1 - h_1$ and $b = y_1 - h_2$, (x, y) will represent the coordinates of the given points when $i = 1, \cdots, n; j = 1, \cdots, m$. Let $\bar{x}$ and $\bar{y}$ be the midpoints of the given x_i and $y_j[\bar{x} = (x_1 + x_n)/2; \bar{y} = (y_1 + y_m)/2]$. The horizontal and vertical distances from the midpoint divided by the uniform spacing $[(x - \bar{x})/h_1$ and $(y - \bar{y})/h_2]$ are then equal respectively, by virtue of equations (1), to $i - (n + 1)/2$ and $j - (m + 1)/2$.

Let P_{hk} $(h = 0, 1, \cdots, r; k = 0, 1, \cdots, s)$ be a set of polynomials of exact degree r and s in $i - (n + 1)/2$ and $j - (m + 1)/2$, i. e.

$$\begin{aligned} P_{hk} = \alpha_{00}^{hk} &+ \alpha_{10}^{hk}[i - (n + 1)/2] + \alpha_{01}^{hk}[j - (m + 1)/2] \\ &+ \alpha_{11}^{hk}[i - (n + 1)/2][j - (m + 1)/2] + \cdots \\ &+ \alpha_{hk}^{hk}[i - (n + 1)/2]^h[j - (m + 1)/2]^k. \end{aligned} \tag{2}$$

P_{00} will always be taken equal to one.

If, now, we approximate $f(x, y)$ by a polynomial of the form

$$\sum_{h=0}^{s}\sum_{k=0}^{r} b_{kh} P_{kh}$$

and solve for the constants b_{kh} in the sense of least squares, we obtain $(r + 1)(s + 1)$ normal equations with the augmented matrix

$$\begin{bmatrix} \sum\sum P_{00} & \sum\sum P_{01} \cdots \sum\sum P_{0s} & \sum\sum P_{10} \cdots \sum\sum P_{1s} \cdots \sum\sum P_{rs} & \sum\sum f(x_i, y_j) \\ \sum\sum P_{00}P_{01} & \sum\sum P_{01}^2 \cdots \sum\sum P_{0s}P_{01} \cdots & \sum\sum P_{rs}P_{01} & \sum\sum f(x_i, y_j)P_{01} \\ \vdots & \vdots & & \\ \sum\sum P_{00}P_{0s} & \sum\sum P_{01}P_{0s} \cdots \sum\sum P_{0s}^2 \cdots & \sum\sum P_{rs}P_{0s} & \sum\sum f(x_i, y_j)P_{0s} \\ \sum\sum P_{00}P_{rs} & \sum\sum P_{01}P_{rs} \cdots \sum\sum P_{0s}P_{rs} \cdots & \sum\sum P_{rs}^2 & \sum\sum f(x_i, y_j)P_{rs} \end{bmatrix}$$

where the notation $\sum \sum P_{hk}P_{cd}$ means

$$\sum_{j=1}^{m} \sum_{i=1}^{n} P_{hk}[i - (n + 1)/2, j - (m + 1)/2]P_{cd}[i - (n + 1)/2, j - (m + 1)/2].$$

If the α_{ij}^{hk} in (2) are chosen so that the polynomials P_{hk} are biorthogonal, i.e.

$$\sum_{j=1}^{m} \sum_{i=1}^{n} P_{hk}P_{cd} = 0 \qquad \text{if } h \neq c \text{ or } k \neq d,$$

the coefficient matrix reduces to a diagonal matrix. The b_{hk} can then be written as

$$b_{hk} = \sum_j \sum_i f(x_i, y_j)P_{hk} / \sum_j \sum_i P_{hk}^2 . \tag{3}$$

As in the univariate case [2], testing the appropriatness of the representation is also facilitated by the use of orthogonal polynomials. Should a polynomial of higher degree be desired, the coefficients b_{hk} need not be recalculated.

3. Constructing Biorthogonal Polynomial Tables. In approximating with orthogonal functions, many of the calculations necessary are independent of the data [for example, the denominator in (3)] and need not be recalculated each time a different set of data is to be fitted. Extensive tables are available for univariate orthogonal polynomials [1], [2]. Their use makes the actual calculation of the coefficients quite easy.

By appropriately choosing the bivariate biorthogonal polynomials, univariate tables that are already available can easily be modified for use in the bivariate case.

THEOREM. *Let P_h $(h = 0, 1, \cdots, r)$ and P_k $(k = 0, 1, \cdots, s)$ be sets of orthogonal polynomials in $(i - (n + 1)/2)$ and $(j - (m + 1)/2)$ respectively with $P_0 = 1$. Let $P_{hk} = P_hP_k$. P_{hk} $(h = 0, 1, \cdots, r; k = 0, 1, \cdots, s)$ is then a set of bivariate biorthogonal polynomials in $(i - (n + 1)/2)$ and $(j - (m + 1)/2)$.*

Proof. It follows from the definition that P_{hk} will be a bivariate polynomial of degree h and k in $(i - (n + 1)/2)$ and $(j - (m + 1)/2)$ respectively.

Biorthogonality of the bivariate polynomials follows from the orthogonality of the univariate polynomials and the fact that

$$\sum_j \sum_i P_{hk}P_{cd} = \sum_j \sum_i P_hP_kP_cP_d = \sum_j P_kP_c \sum_i P_hP_d .$$

Taking $P_{hk} = P_hP_k$, it is possible to utilize, in bivariate polynomial approximation, tables that were constructed for use in the univariate case. In general, if one regards the rectangular array of values for a given n in Fisher and Yates statistical tables [2] as a matrix, then the corresponding matrix for bivariate biorthogonal polynomials would be a Kronecker product [5] of corresponding matrices.

For example, from the entries for $n = 3$ and $n = 4$ in the Fisher and Yates tables [2], one gets as the entry in the bivariate table for $n = 3$, $m = 4$

i	j	P_{01}	P_{02}	P_{03}	P_{10}	P_{11}	P_{12}	P_{13}	P_{20}	P_{21}	P_{22}	P_{23}
1	1	−3	+1	−1	−1	+3	−1	+1	+1	−3	+1	−1
1	2	−1	−1	+3	−1	+1	+1	−3	+1	−1	−1	+3
1	3	+1	−1	−3	−1	−1	+1	+3	+1	+1	−1	−3
1	4	+3	+1	+1	−1	−3	−1	−1	+1	+3	+1	+1
2	1	−3	+1	−1	0	0	0	0	−2	+6	−2	+2
2	2	−1	−1	+3	0	0	0	0	−2	+2	+2	−6
2	3	+1	−1	−3	0	0	0	0	−2	−2	+2	+6
2	4	+3	+1	+1	0	0	0	0	−2	−6	−2	−2
3	1	−3	+1	−1	+1	−3	+1	−1	+1	−3	+1	−1
3	2	−1	−1	+3	+1	−1	−1	+3	+1	−1	−1	+3
3	3	+1	−1	−3	+1	+1	−1	−3	+1	+1	−1	−3
3	4	+3	+1	+1	+1	+3	+1	+1	+1	+3	+1	+1
		60	12	60	8	40	8	40	24	120	24	120

where the last line represents the sums of squares of all elements in the particular column. All entries in the bivariate table are products of corresponding entries in the univariate table.

Bivariate tables constructed as outlined here could be used in exactly the same manner as recommended for univariate tables [see 2].

4. Extension to n Variables. The extension to n variables is straightforward. The n dimension analogue to equation (3) is

$$b_{h_1 \cdots h_n} = \sum_{i_n=1}^{m_n} \cdots \sum_{i_1=1}^{m_1} f(x_{i_1}^{(1)}, x_{i_2}^{(2)}, \cdots, x_{i_n}^{(n)}) P_{h_1 \cdots h_n} \Big/ \sum_{i_n=1}^{m_n} \cdots \sum_{i_1=1}^{m_1} P_{h_1 \cdots h_n}^2$$

where the $P_{h_1 \cdots h_n}$ are orthogonal polynomials in n variables and the $b_{h_1 \cdots h_n}$ are the coefficients in the approximation

$$f(x^{(1)}, \cdots, x^{(n)}) \approx \sum_{h_n=0}^{r_n} \cdots \sum_{h_1=0}^{r_1} b_{h_1 \cdots h_n} P_{h_1 \cdots h_n} \Big/ \sum_{i_n=1}^{m_n} \cdots \sum_{i_1=1}^{m_1} P_{h_1 \cdots h_n}^2$$

so as to minimize

$$\sum_{i_n=1}^{m_n} \cdots \sum_{i_1=1}^{m_1} \left[f(x_{i_1}^{(1)}, \cdots, x_{i_n}^{(n)}) - \sum_{h_n=0}^{r_n} \cdots \sum_{h_1=0}^{r_1} b_{h_1 \cdots h_n} P_{h_1 \cdots h_n} \right]^2.$$

As in the bivariate case, one can make use of univariate tables in multivariate polynomial approximation by taking for the multivariate polynomials the products of corresponding univariate orthogonal polynomials.

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